

Mutual friction in helium II: a microscopic approach

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We develop a microscopic model of mutual friction represented by the dissipative dynamics of a normal fluid flow which interacts with the helical normal modes of vortices comprising a lattice in thermal equilibrium. Such vortices are assumed to interact with the quasiparticles forming the normal fluid through a pseudomomentum-conserving scattering Hamiltonian. We study the approach to equilibrium of the normal fluid flow for temperatures below 1 K, deriving an equation of motion for the quasiparticle pseudomomentum which leads to the expected form predicted by the HVBK equations. We obtain an expression for the mutual friction coefficient B in terms of microscopic parameters, which turns out to be practically independent of the vortex mass for values arising from diverse theories. By comparing our expression of B with previous theoretical estimates, we deduce interesting qualitative features about the excitation of Kelvin modes by the quasiparticle scattering.

I. INTRODUCTION

When a sufficiently fast rotating sample of liquid helium is cooled below the lambda temperature, all the rotation of the superfluid becomes concentrated in a uniform array of quantized vortex filaments parallel to the axis of rotation [1, 2]. By contrast, the macroscopic superfluid velocity field, corresponding to spatial averages over regions large compared with the spacing between vortices, yields the usual configuration of solid body flow, $\mathbf{v}_s(\mathbf{r}) = \Omega_{\text{rot}} \hat{\mathbf{z}} \times \mathbf{r}$ for a rotation frequency Ω_{rot} around the z axis. Just as the superfluid flow is microscopically formed by vortices, the normal fluid consists of superfluid quasiparticle excitations, phonons and rotons, the average flow of which is characterized by the normal fluid velocity field \mathbf{v}_n . In equilibrium both fluids move with the same velocity $\mathbf{v}_n = \mathbf{v}_s$ and such a behavior arises, from a microscopic viewpoint, from the vortex motion with the normal fluid velocity in order to avoid dissipation. That is, any relative motion of vortices with respect to the normal fluid in their vicinity, is subjected to a friction force that causes such a motion to eventually cease. Such a *mutual friction force* [3] between the two fluids plays then a central role in the mechanism which maintains the stability of the above equilibrium state. A well-known phenomenological model for this macroscopic dynamics is represented by the so-called Hall-Vinen-Bekharevich-Khalatnikov (HVBK) equations [3, 4], basically consisting of a Navier-Stokes equation for the normal fluid and an Euler equation for the superfluid, which, in the absence of pressure and temperature gradients, are coupled together only by mutual friction. There is a simple configuration which allows to show the basic features of this process, namely rectilinear flows of uniform vorticity [5],

$$\mathbf{v}_s(\mathbf{r}, t) = -2\Omega_s(t) y \hat{\mathbf{x}} \quad (1.1)$$

$$\mathbf{v}_n(\mathbf{r}, t) = -2\Omega_n(t) y \hat{\mathbf{x}} \quad (1.2)$$

($y < 0$), where $\Omega_s(t)$ and $\Omega_n(t)$ should converge for $t \rightarrow \infty$ to a steady state value. Then, to make contact with the standard rotational configuration, we may identify such a value with the former angular velocity Ω_{rot} . Note that this assignment leads to a uniform vorticity $\nabla \times \mathbf{v}_s = 2\Omega_{\text{rot}} \hat{\mathbf{z}}$ which coincides with that of the rotational scheme. The HVBK equations for such flows are very simple and read

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} = \mathbf{F} = -\rho_s \frac{\partial \mathbf{v}_s}{\partial t}, \quad (1.3)$$

where ρ_n and ρ_s denote the normal fluid and superfluid mass densities, respectively, and the mutual friction force \mathbf{F} can be written for temperatures below 1 K as [5],

$$\mathbf{F} = -B\rho_n \Omega_{\text{rot}}(\mathbf{v}_n - \mathbf{v}_s), \quad (1.4)$$

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being B a dimensionless dissipative coefficient. Such a low temperature regime corresponds to $\rho_n \ll \rho_s$, which, according to (1.3), implies that the main time dependence should lie within the normal fluid velocity. This suggests that a suitable approach to the problem may consist in regarding the superfluid component as a thermal equilibrium heat bath which interacts with a nonequilibrium normal fluid flow. Keeping such a picture as our basic premise, we shall analyze in the present paper a microscopic model of mutual friction, which reproduces the main features of the above macroscopic dynamics, yielding an explicit expression of B as a function of microscopic parameters.

The microscopic basis of mutual friction remains as one of the most intricate problems of superfluidity. In such a context, theoretical approaches may strongly differ, even in significant questions such as the existence of a nondissipative component of the mutual friction force [6], which is absent from our modelling [5]. A better understanding of the microscopic principles governing mutual friction would also contribute to clarify important issues on the subject of quantum turbulence at finite temperature [7]. In fact, it is just the mutual friction force which accounts for the strong locking between superfluid and normal fluid along the turbulent cascade, where recent simulations have shown that the residual slip velocity $\mathbf{v}_s - \mathbf{v}_n$ plays a central role [8]. In addition, such simulations suggest that the cross-over between zero-temperature and finite temperature quantum turbulence occurs at a lower temperature than the usual estimation of 1 K, hence partially placing the latter regime within the temperature range of the present investigation.

Another important source of controversy arises from the mass of quantized vortices. On the one hand, many works have considered it as a negligible parameter under the assumption that it should be equivalent to the hydrodynamic mass of a core of atomic dimensions [1]. Another theories, however, yield several orders of magnitude higher values for the vortex mass, casting doubt on models based on massless vortices [9]. Moreover, it has been argued that an unambiguous vortex mass may not exist, and that inertial effects in vortex dynamics may be scenario-dependent [10]. Finally, we should also mention that there have been conflicting results for the vortex mass in superconductors as well [11]. A possible way out to such uncertainties has been recently suggested based on the concept of *pseudomomentum*. In fact, just as the momentum corresponds to the generator of ordinary translations, the so-called pseudomomentum generates translations which keep an eventual background medium fixed [12]. Thus, the pseudomomentum often appears as a useful tool in fluid mechanics [13]. In the present case we shall concentrate our attention on the pseudomomentum of quasiparticles forming the normal fluid and on the pseudomomentum of vortices, being the background medium the superfluid. The quasiparticle pseudomomentum per unit volume turns out to be simply $\rho_n(\mathbf{v}_n - \mathbf{v}_s)$, which may be easily interpreted as the ordinary momentum of a normal fluid moving relatively to the superfluid. On the other hand, the vortex pseudomomentum requires a more careful treatment, recently shown to lead to an alternative approach to the dissipative vortex dynamics, free of the ambiguities carried by the uncertainty of the vortex mass [5]. Our present study of the dissipative normal fluid dynamics will utilize the concept of pseudomomentum as an important tool, finding again results which are practically independent of the vortex mass for a wide range of values.

The dynamics of the normal fluid flow is far less known than that of the superfluid, because, apart from very recent efforts [14], there is a lack of experimental observation techniques at these low temperatures. Only recently, theoretical research has been focused on this issue by means of numerical simulations, which provided valuable information about normal fluid flow patterns due to the mutual friction interaction [15]. Another interesting aspect, which has been recently investigated, concerns the stability characteristics of the normal fluid flow under mutual friction forcing from the superflow [16].

Our approach will consist in assuming a heat bath formed by a vortex lattice in thermal equilibrium, which interacts with a quasiparticle flow. The dominant contribution to the heat capacity of such a lattice should arise from the thermal excitation of helical waves, corresponding to effectively independent vortices [17]. The role of such oscillations in mutual friction has been scarcely treated in the literature. We are only able to mention a couple of papers [18, 19], that long ago reached the conclusion that the damping of vortex oscillations due to phonon scattering, should not modify appreciably the value of the friction coefficient calculated for a rigid vortex. The same conclusion was recently obtained for a high-frequency branch of helical waves, within a wider temperature range, including a roton-dominated regime [20].

In building a theory with massive vortices, one can readily make use of a close analogy with the well-known electrodynamical problem of a point charge subjected to magnetic and electric fields [21]. Particularly, the quantization of the theory, which greatly simplifies the treatment when the scattering excitation of vortex waves is taken into account [18], arises immediately from this analogy. Such an analogy also leads to an immediate identification of the vortex pseudomomentum, allowing us to build a proper form for a pseudomomentum-conserving scattering Hamiltonian.

This paper is organized as follows. In the next section, we propose a Hamiltonian model for the interaction of a vortex line with a quasiparticle gas and a background superflow of uniform vorticity. In Sec. III we obtain the equation of motion for the quasiparticle pseudomomentum, which leads to the expected form predicted by the HVBK equations. In Sec. IV we study our expression for the mutual friction coefficient B comparing this result with previous theoretical estimates. Finally in Sec. V we give our concluding remarks.

II. HAMILTONIAN MODEL

A. Vortex Hamiltonian

Let us consider an otherwise rectilinear vortex filament performing helical oscillations about its unperturbed position parallel to the z axis [1]. The wavelength λ is supposed to be much greater than the amplitude (radius of the helix), and to have a full description of the helix, one should also know the direction (right or left) of the helical deformation, or equivalently, the direction of the wave vector $k\hat{\mathbf{z}}$ ($k = \pm 2\pi/\lambda$). Thus, it suffices to know the location of the vortex core at a given plane normal to the z axis, say $z = 0$, to have a full determination of the position of the whole vortex filament. Periodic boundary conditions over a length L (vortex line length) along the z axis determine the possible values of the wave vector as $k = 2\pi m/L$, where m is an integer. The vortex core position \mathbf{r} may then be written as a summation over generalized two-dimensional coordinates \mathbf{r}_k associated to normal modes labeled by the wave vector $k\hat{\mathbf{z}}$. We shall first consider true oscillatory modes ($k \neq 0$), since modes with $k = 0$ corresponding to rigid displacements of the vortex filament require a separate treatment [5]. The equation of motion for $\mathbf{r}_k(t)$ is given by [1, 20]:

$$\ddot{\mathbf{r}}_k = \Omega \hat{\mathbf{z}} \times \dot{\mathbf{r}}_k - \Omega \omega_k \mathbf{r}_k, \quad (2.1)$$

with

$$\Omega = \rho_s \kappa / m_v \quad (2.2)$$

and

$$\omega_k = \frac{\kappa k^2}{4\pi} [-\ln(|k|a) + 0.116], \quad (2.3)$$

where m_v denotes the vortex mass per unit length and $\kappa = h/m_4$ denotes the quantum of circulation, given by the ratio of Planck's constant and the mass of one ^4He atom. The vortex core parameter $a \sim 1 \text{ \AA}$ in (2.3) is assumed to be much less than the wavelength ($|k|a \ll 1$), which in turn ensures that $\omega_k \ll \Omega$ [1, 20]. The first term in equation (2.1) corresponds to the Magnus force, whereas the second term stems from the induced velocity on the vortex line element by the helix curvature. Assuming that the vortex has a counterclockwise circulation, the frequencies (2.2) and (2.3) are positive, and then equation (2.1) turns out to be analogous to that ruling the two-dimensional motion of a negative point charge, in a uniform magnetic field parallel to the z axis and subjected to a harmonic central force. Such an equation derives from the following Hamiltonian:

$$\frac{m_v}{2} (\mathbf{v}_k^2 + \Omega \omega_k \mathbf{r}_k^2), \quad (2.4)$$

being

$$\mathbf{v}_k = \frac{\mathbf{p}_k}{m_v} + \frac{\Omega}{2} \hat{\mathbf{z}} \times \mathbf{r}_k, \quad (2.5)$$

where \mathbf{p}_k denotes the conjugate momentum to \mathbf{r}_k . In fact, from Hamilton equations it is easy to check that (2.5) corresponds to the velocity $\dot{\mathbf{r}}_k$, while the acceleration is indeed given by (2.1). From the above electromagnetic analogy it is also useful to represent the coordinate \mathbf{r}_k as the sum of the center coordinate $\mathbf{R}_0^{(k)}$ of the cyclotron circle plus the relative coordinate \mathbf{R}'_k from such a center [21],

$$\mathbf{r}_k = \mathbf{R}_0^{(k)} + \mathbf{R}'_k = \mathbf{R}_0^{(k)} + \mathbf{v}_k \times \hat{\mathbf{z}} / \Omega. \quad (2.6)$$

Then, from Hamilton equations one easily obtains the following pair of coupled equations for $\mathbf{R}_0^{(k)}$ and \mathbf{R}'_k :

$$\dot{\mathbf{R}}_0^{(k)} = -\omega_k \hat{\mathbf{z}} \times \mathbf{R}_0^{(k)} - \omega_k \hat{\mathbf{z}} \times \mathbf{R}'_k \quad (2.7a)$$

$$\dot{\mathbf{R}}'_k = (\Omega + \omega_k) \hat{\mathbf{z}} \times \mathbf{R}'_k + \omega_k \hat{\mathbf{z}} \times \mathbf{R}_0^{(k)}. \quad (2.7b)$$

The solution of the above system can more simply be expressed in complex notation ($\mathbf{r} = x + iy$), and to first order in ω_k/Ω we have,

$$\mathbf{R}_0^{(k)} = A_1 \exp(-i\omega_k t) - \frac{\omega_k}{\Omega} A_2 \exp[i\Omega(1 + \omega_k/\Omega)t] \quad (2.8a)$$

$$\mathbf{R}'_k = A_2 \exp[i\Omega(1 + \omega_k/\Omega)t] - \frac{\omega_k}{\Omega} A_1 \exp(-i\omega_k t), \quad (2.8b)$$

where A_1 and A_2 are complex numbers depending on initial conditions. Then, neglecting corrections of order ω_k/Ω , the solution corresponds to a decoupling of the system (2.7):

$$\dot{\mathbf{R}}_0^{(k)} = -\omega_k \hat{\mathbf{z}} \times \mathbf{R}_0^{(k)} \quad (2.9a)$$

$$\dot{\mathbf{R}}'_k = \Omega \hat{\mathbf{z}} \times \mathbf{R}'_k, \quad (2.9b)$$

that is, we shall simply have circular trajectories, namely for $\mathbf{R}_0^{(k)}$ a clockwise one with angular frequency ω_k and for \mathbf{R}'_k a counterclockwise one with angular frequency Ω . Then, it becomes clear that $\mathbf{R}_0^{(k)}$ should be ascribed to Kelvin modes, while \mathbf{R}'_k should correspond to the cyclotron ones [1, 20]. Theoretical estimates of the vortex mass [1, 9] lead to cyclotron frequency values in (2.2) of the order or greater than $k_B T/\hbar$ for $T < 1$ K. This seems to indicate that quantum effects could be of importance. Quantization of coordinate and momentum arises straightforwardly from the electromagnetic analogy and reads [21],

$$\mathbf{R}_0^{(k)} = \sqrt{\frac{\hbar}{2\rho_s \kappa L}} [e^{-ikz} \beta_k^\dagger (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + e^{ikz} \beta_k (\hat{\mathbf{x}} - i\hat{\mathbf{y}})] \quad (2.10)$$

$$\mathbf{R}'_k = \sqrt{\frac{\hbar}{2\rho_s \kappa L}} [e^{-ikz} \alpha_k (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + e^{ikz} \alpha_k^\dagger (\hat{\mathbf{x}} - i\hat{\mathbf{y}})] \quad (2.11)$$

$$\mathbf{p}_k = i\sqrt{\frac{\hbar\rho_s \kappa}{8L}} [e^{-ikz} (\beta_k^\dagger - \alpha_k) (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + e^{ikz} (\alpha_k^\dagger - \beta_k) (\hat{\mathbf{x}} - i\hat{\mathbf{y}})], \quad (2.12)$$

where α_k^\dagger (β_k^\dagger) denotes a creation operator of right (left) circular quanta. The z dependence of coordinates and momentum corresponds to the rotation generated on following the helix path. Normal modes corresponding to $k\hat{\mathbf{z}}$ are ruled by a Hamiltonian of the form (2.4):

$$H_k = \int_0^L dz \frac{m_v}{2} [\mathbf{v}_k^2 + \Omega \omega_k \mathbf{r}_k^2] \quad (2.13)$$

where, taking into account (2.10) to (2.12), one may verify that \mathbf{v}_k^2 and \mathbf{r}_k^2 do not depend on z and also that the limit $\omega_k/\Omega \ll 1$ yields [20],

$$H_k = \hbar\Omega(\alpha_k^\dagger \alpha_k + \frac{1}{2}) + \hbar\omega_k(\beta_k^\dagger \beta_k + \frac{1}{2}). \quad (2.14)$$

That is, both polarizations (cyclotron and Kelvin modes) become decoupled, as seen from a classical viewpoint in equations (2.9). Here it is instructive to evaluate the limit of a massless vortex, which is often found in the literature [17]. In fact, the limit $\Omega \rightarrow \infty$ in (2.1) transforms such an equation into a first-order one, with the consequence that the Cartesian components x_k and y_k of \mathbf{r}_k become conjugate variables. Note that this amounts to ignoring the cyclotron motion by setting $\mathbf{r}_k \equiv \mathbf{R}_0^{(k)}$ (cf. Eq. (2.6)), where the components of $\mathbf{R}_0^{(k)}$ in (2.10) obey canonical commutation relations.

The Hamiltonian of the $k = 0$ modes reads [5]

$$H_0 = \frac{m_v L}{2} (v_{0x}^2 + v_{0y}^2) - \Omega_{\text{rot}} \rho_s \kappa L y_0^2 \quad (2.15)$$

with $v_{0x} = p_{0x}/m_v - \Omega y_0$, $v_{0y} = p_{0y}/m_v$. The quantization of the coordinate $\mathbf{r}_0 = \mathbf{R}_0^{(0)} + \mathbf{R}'_0 = \mathbf{R}_0^{(0)} + \mathbf{v}_0 \times \hat{\mathbf{z}}/\Omega$ remains given through the previous expressions (2.10) and (2.11), whereas the momentum expression (2.12) becomes changed according to the Landau gauge as $\mathbf{p}_0 = \sqrt{\hbar\rho_s \kappa/(2L)} [i(\beta_0^\dagger - \beta_0)\hat{\mathbf{x}} + (\alpha_0^\dagger + \alpha_0)\hat{\mathbf{y}}]$. The Hamiltonian (2.15) can be exactly solved [21] yielding in the limit $\Omega_{\text{rot}} \ll \Omega$ the decoupling of cyclotron and translational modes:

$$H_0 = \hbar\Omega(\alpha_0^\dagger \alpha_0 + \frac{1}{2}) + \frac{\hbar\Omega_{\text{rot}}}{2}(\beta_0^\dagger - \beta_0)^2. \quad (2.16)$$

Thus, the Heisenberg equation of motion for the cyclotron coordinate \mathbf{R}'_0 is given again by (2.9b), whereas the translational coordinate evolves according to the superfluid velocity field

$$\dot{\mathbf{R}}_0^{(0)} = -2\Omega_{\text{rot}} Y_0^{(0)} \hat{\mathbf{x}}. \quad (2.17)$$

In conclusion, the vortex Hamiltonian may be written as follows:

$$H_v = \sum_k \left[\hbar\Omega \left(\alpha_k^\dagger \alpha_k + \frac{1}{2} \right) + \hbar\omega_k \left(\beta_k^\dagger \beta_k + \frac{1}{2} \right) \right] + \frac{\hbar\Omega_{\text{rot}}}{2} (\beta_0^\dagger - \beta_0)^2 \quad (2.18)$$

with $k = 2\pi m/L$ and m an integer.

B. Quasiparticle Hamiltonian

The normal fluid will be represented by the following Hamiltonian:

$$H_n = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}}, \quad (2.19)$$

where $a_{\mathbf{q}}^{\dagger}$ denotes a creation operator of quasiparticle excitations of pseudomomentum $\hbar \mathbf{q}$ and frequency $\omega_{\mathbf{q}}$. Such a frequency corresponds to the lab frame and may be written as a Doppler-shifted frequency from the superfluid frame, $\omega_{\mathbf{q}} = \omega_q + \mathbf{q} \cdot \mathbf{v}_s$, where ω_q is the familiar (isotropic) dispersion relationship of ^4He quasiparticle excitations. Now, the background superflow velocity should be much less than the Landau critical velocity ~ 60 m/s, so we may safely approximate $\omega_{\mathbf{q}} = \omega_q$ in (2.19). Note also that we disregard any interaction between the quasiparticles themselves, since we shall work at low enough temperature, so that they remain dilute allowing their treatment as a noninteracting gas.

C. Interaction Hamiltonian

The interaction Hamiltonian between the vortex and the quasiparticles will be represented by the pseudomomentum-conserving form:

$$H_{\text{int}} = \sum_{\mathbf{p}, \mathbf{q}} \int_0^L dz \Lambda_{\mathbf{p}\mathbf{q}} a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}} \exp[-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{r}(z) - i(p_z - q_z)z], \quad (2.20)$$

where $\mathbf{r}(z) = \mathbf{R}_0(z) + \mathbf{R}'(z)$, being $\mathbf{R}_0(z) = \sum_k \mathbf{R}_0^{(k)}(z)$ and $\mathbf{R}'(z) = \sum_k \mathbf{R}'_k(z)$, with $\mathbf{R}_0^{(k)}(z)$ and $\mathbf{R}'_k(z)$ given by (2.10) and (2.11), respectively. The parameters $\Lambda_{\mathbf{p}\mathbf{q}}$ in (2.20) represent scattering amplitudes depending on wave vectors of scattered quasiparticles. The vortex pseudomomentum per unit length [5] $\mathbf{K}(z) = -\rho_s \kappa \hat{\mathbf{z}} \times \mathbf{R}_0(z)$ integrated along the vortex line yields the generator of vortex translations or vortex pseudomomentum $\int_0^L \mathbf{K}(z) dz = -\rho_s \kappa L \hat{\mathbf{z}} \times \mathbf{R}_0^{(0)}$, which involves only translational coordinates, as expected. Then, adding such a pseudomomentum to the quasiparticle pseudomomentum $\sum_{\mathbf{q}} \hbar \mathbf{q} a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}}$, we have the pseudomomentum of the whole system, which can be shown to commute with H_{int} . Note that only the x -component of the vortex pseudomomentum $\rho_s \kappa L Y_0^{(0)}$ will commute with the vortex Hamiltonian (2.18), unless $\Omega_{\text{rot}} = 0$. This result may be easily interpreted, since a superflow of velocity $\mathbf{v}_s = -2\Omega_{\text{rot}} y \hat{\mathbf{x}}$ produces a translation symmetry breaking in the y -direction.

III. EQUATION OF MOTION FOR THE NORMAL FLUID FLOW

The interaction Hamiltonian (2.20) is difficult to deal with, so recalling the low amplitude of the helical oscillations, we may rewrite the exponential in (2.20) as

$$\begin{aligned} \exp[-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{r}(z) - i(p_z - q_z)z] = \\ \exp[-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{R}_0^{(0)}] \exp[-i(p_z - q_z)z] \exp[-i(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{r}(z) - \mathbf{R}_0^{(0)})] \end{aligned} \quad (3.1)$$

and next approximate the last exponential to first order in $\mathbf{r}(z) - \mathbf{R}_0^{(0)}$. This procedure, however, is not valid for vortex modes with frequencies approaching zero, i.e. the lowest part of Kelvin's spectrum, as noted early by Fetter [18]. In fact, he showed that retaining a finite number of terms of such an exponential expansion leads to divergent results, analogous to those of the "infrared catastrophe" in electrodynamics. Here it is expedient to recall that within our study, each vortex forms part of a vortex lattice which will be regarded as a heat bath for the quasiparticle flow. Now, it is well known that rather simple models of heat bath often provide suitable descriptions of realistic environments [22]. Relying on this hypothesis and to overcome the above difficulty, we shall represent Kelvin's spectrum by a single frequency w_0 , which will be eventually regarded as a temperature-dependent parameter in order to take into account the distinct features of the interaction of such waves with phonons and rotons. In summary, we shall make use of a simplified model of heat bath consisting of vortex modes of two frequencies ($w_0 \ll \Omega$) of opposite polarization. On the other hand, neglecting the vortex displacement in the y -direction [5], we shall replace the translational coordinate operator in the first exponential on the right-hand side of Eq. (3.1) by the c -number $\langle \mathbf{R}_0^{(0)} \rangle = -2\Omega_{\text{rot}} \langle Y_0^{(0)} \rangle t \hat{\mathbf{x}}$ (cf. Eq. (2.17)). Such a replacement becomes equivalent to having time-dependent scattering amplitudes in Eq. (2.20), i.e. with a

time dependent phase factor, $\Lambda_{\mathbf{p}\mathbf{q}} \exp[i(p_x - q_x)2\Omega_{\text{rot}} \langle Y_0^{(0)} \rangle t]$. This factor, however, would not have any practical incidence, since our results will be shown to be dependent upon the absolute value of the scattering amplitudes. Finally, taking into account these approximations we may replace $\exp[-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{r}(z)] \simeq 1 - i(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{r}(z) - \mathbf{R}_0^{(0)})$ in Eq. (2.20) yielding [20]

$$H_{\text{int}} = \sqrt{\frac{\hbar L}{2\rho_s \kappa}} \sum_{k, \mathbf{p}, \mathbf{q}} \delta_{p_z q_z} \Lambda_{\mathbf{p}\mathbf{q}} a_{\mathbf{p}}^\dagger a_{\mathbf{q}} \{ [(q_y - p_y) + i(q_x - p_x)][\alpha_k^\dagger + (1 - \delta_{k0})\beta_k] \\ + [(p_y - q_y) + i(p_x - q_x)][\alpha_k + (1 - \delta_{k0})\beta_k^\dagger] \}. \quad (3.2)$$

There is an additional parameter to be taken into account in our vortex heat bath, that is the total number of modes $2L/\lambda_{\text{min}}$, where the factor 2 in front of the expression comes from the two possible signs for k . We shall assume for simplicity that both polarizations have a common ‘ultraviolet’ cutoff λ_{min} , which according to Sec. II A, should be greater than the vortex core parameter ($\sim 1 \text{ \AA}$) and the mean radius of the helix. Such a radius turns out to be of the order of the core parameter for cyclotron modes, while for Kelvin modes in a lattice of $\Omega_{\text{rot}} \sim 1 \text{ s}^{-1}$ has been estimated [17] as $\sim 10^3 \text{ \AA} \sqrt{T/K}$. So we shall assume $\lambda_{\text{min}} \sim 10^3 \text{ \AA}$ in our calculations.

A total Hamiltonian given by the sum of (2.18), (2.19) and (3.2) yields dissipative evolutions for each vortex mode, which for cyclotron modes are ruled by [20]:

$$\langle \ddot{\mathbf{R}}'_k \rangle = \Omega \left(1 - \frac{D'}{\rho_s \kappa} \right) \hat{\mathbf{z}} \times \langle \dot{\mathbf{R}}'_k \rangle - \Omega \frac{D}{\rho_s \kappa} \langle \dot{\mathbf{R}}'_k \rangle, \quad (3.3)$$

where explicit expressions for the longitudinal and transverse friction coefficients, D and D' , respectively, can be found in [23]. There we have shown that a scattering amplitude given by [24]

$$\Lambda_{\mathbf{p}\mathbf{q}} = \frac{\hbar \kappa}{V c_s} \sqrt{\frac{19}{140} |\omega'_p| |\omega'_q|}, \quad (3.4)$$

where V denotes the volume of the system and ω'_p (c_s) denotes the quasiparticle group (sound) velocity, leads to a very good agreement with the experimental determinations of the longitudinal friction coefficient for temperatures below 1 K. The presence of the viscous force represented by the last term in (3.3) leads to a vanishing asymptotic velocity, $\langle \dot{\mathbf{R}}'_k \rangle_{t \rightarrow \infty} = 0$, and the same behavior will present the coordinate, $\langle \mathbf{R}'_k \rangle_{t \rightarrow \infty} = 0$.

To summarize, we have focused in previous works on the dynamics of an individual vortex line which interacts with a quasiparticle heat bath. Now, being focused on the time evolution of the normal fluid, we shall concentrate on the dissipative dynamics of a quasiparticle flow which interacts with the heat bath formed by a uniform array of N_v quantized vortex filaments. We have derived in Appendix A the system (A6) of non-Markovian equations that rule the time evolution of the quasiparticle populations $n_{\mathbf{q}}$. Then, from (A6) one easily obtains the following equation of motion for the quasiparticle pseudomomentum:

$$\sum_{\mathbf{q}} \hbar \mathbf{q} \dot{n}_{\mathbf{q}} = \frac{2L^2 N_v}{\rho_s \kappa \lambda_{\text{min}}} \sum_{\mathbf{p}, \mathbf{q}, i} |\Lambda_{\mathbf{p}\mathbf{q}}|^2 \delta_{p_z q_z} (\mathbf{p} - \mathbf{q})(\mathbf{p} - \mathbf{q})^2 \int_0^t d\tau \cos[(\omega_p - \omega_q + w_i)\tau] \{ n_{\mathbf{q}}(t - \tau) \\ \times [1 + n_{\mathbf{p}}(t - \tau)] + [n_{\mathbf{q}}(t - \tau) - n_{\mathbf{p}}(t - \tau)] [e^{\hbar w_i / k_B T} - 1]^{-1} \}, \quad (3.5)$$

where w_i denotes the frequencies Ω and w_0 . We shall assume that the quasiparticle numbers in the above expression are well described by a local equilibrium form:

$$n_{\mathbf{q}} = [e^{\hbar \omega_{\mathbf{q}} / k_B T} - 1]^{-1} \simeq [e^{\hbar \omega_q / k_B T} - 1]^{-1} + \frac{\hbar \mathbf{q} \cdot [\mathbf{v}_n - \mathbf{v}_s]}{4k_B T \sinh^2(\hbar \omega_q / 2k_B T)}, \quad (3.6)$$

where the quasiparticle frequency is measured from a reference frame where the local normal fluid is at rest, $\omega_{\mathbf{q}} = \omega_q - \mathbf{q} \cdot [\mathbf{v}_n - \mathbf{v}_s]$. Thus, the quasiparticle pseudomomentum $\sum_{\mathbf{q}} \hbar \mathbf{q} n_{\mathbf{q}} = A L \rho_n (\mathbf{v}_n - \mathbf{v}_s)$ corresponds to a macroscopically small area A of the x - y plane containing N_v vortices, where the spatial dependence of the fields \mathbf{v}_n and \mathbf{v}_s can be neglected. Then, assuming a time dependence stemming exclusively from $\mathbf{v}_n(t)$ (cf. Sec. I), a straightforward calculation leads to the following non-Markovian equation:

$$\dot{\mathbf{v}}_n = - \int_0^t d\tau [\mathbf{v}_n(t - \tau) - \mathbf{v}_s] \mu(\tau), \quad (3.7)$$

with a memory kernel given by

$$\begin{aligned} \mu(\tau) = & \frac{L \hbar \Omega_{\text{rot}}}{k_B T \rho_n \rho_s \kappa^2 \lambda_{\min}} \sum_{\mathbf{p}, \mathbf{q}, i} |\Lambda_{\mathbf{p}\mathbf{q}}|^2 \delta_{p_z q_z} \cos[(\omega_p - \omega_q + w_i)\tau] \{ |\mathbf{p} - \mathbf{q}|^4 g_+(w_i) \\ & + [|\hat{\mathbf{z}} \times (\mathbf{p} \times \hat{\mathbf{z}})|^4 - |\hat{\mathbf{z}} \times (\mathbf{q} \times \hat{\mathbf{z}})|^4] g_-(w_i) \} n(\omega_q) [1 + n(\omega_p)] [1 + n(w_i)], \end{aligned} \quad (3.8)$$

where

$$g_{\pm}(w_i) = \frac{n(\omega_p)}{n(\omega_q - w_i)} \pm \frac{n(\omega_q) \exp[\hbar(\omega_q - \omega_p - w_i)/k_B T]}{n(\omega_p + w_i)} \quad (3.9)$$

and $n(w) = [\exp(\hbar w/k_B T) - 1]^{-1}$. In the thermodynamic limit, the summations over \mathbf{p} and \mathbf{q} in (3.8) become integrals and $\mu(\tau)$ acquires a finite lifetime. If such a lifetime can be regarded as microscopic in comparison with the observational timescale, equation (3.7) may be transformed according to the Markov approximation into the differential equation:

$$\dot{\mathbf{v}}_n = -\nu[\mathbf{v}_n(t) - \mathbf{v}_s] \quad (3.10)$$

with

$$\begin{aligned} \nu = & \int_0^\infty \mu(\tau) d\tau = \frac{L \hbar \Omega_{\text{rot}}}{k_B T \rho_n \rho_s \kappa^2 \lambda_{\min}} \sum_{\mathbf{p}, \mathbf{q}, i} |\Lambda_{\mathbf{p}\mathbf{q}}|^2 \delta_{p_z q_z} \delta(\omega_p - \omega_q + w_i) \\ & \times |\mathbf{p} - \mathbf{q}|^4 n(\omega_q) [1 + n(\omega_p)] [1 + n(w_i)], \end{aligned} \quad (3.11)$$

where the continuum limit corresponds to the replacement $\sum_{\mathbf{p}, \mathbf{q}} \delta_{p_z q_z} \rightarrow [A^2 L / (2\pi)^5] \int d^3 \mathbf{p} \int d^3 \mathbf{q} \delta(p_z - q_z)$. Actually, we have studied in Appendix B the non-Markovian equation (3.7) finding that memory effects are negligible for $\Omega_{\text{rot}} \ll w_0$, which will be assumed hereafter. Finally, taking into account (3.10), (1.3) and (1.4), we may obtain the expression of the mutual friction parameter from $B = \nu/\Omega_{\text{rot}}$.

IV. STUDY OF THE MUTUAL FRICTION PARAMETER B

An explicit expression for the friction parameter B can be extracted by computing the right-hand side of equation (3.11) (see Appendix B):

$$B = \frac{19\hbar^3}{70(2\pi)^2 c_s^2 \rho_n \rho_s k_B T \lambda_{\min}} \sum_i [1 + n(w_i)] \int_0^\infty dp |\omega'_p| n(\omega_p + w_i) [1 + n(\omega_p)] \sum_j \Gamma(p, q_j^{(i)}), \quad (4.1)$$

where

$$\Gamma(p, q) = \begin{cases} p^2 q (q^4 + p^4/5 + 2p^2 q^2) & (p \leq q) \\ q^2 p (p^4 + q^4/5 + 2p^2 q^2) & (q \leq p) \end{cases} \quad (4.2)$$

and $q = q_j^{(i)}$ denote the roots of the equation $\omega_q = \omega_p + w_i$. From (4.1) we may see that B consists of two terms arising from the frequencies $w_i = w_0$ and $w_i = \Omega$. Such contributions, however, are weighted by respective factors $[1 + n(w_0)] \gg [1 + n(\Omega)]$, so the cyclotron contribution will be always negligible with respect to that arising from the frequency w_0 and we have that, in practice, B will correspond to the limit of massless vortices, $\Omega \rightarrow \infty$. The expression (4.1) leads to simpler phonon and roton approximations. If we restrict ourselves to $w_0 \lesssim 10^{10} \text{ s}^{-1}$, such a frequency may be neglected everywhere in (4.1), except in the factor $[1 + n(w_0)]$. Then, to approximate for phonon temperatures ($T < 0.4 \text{ K}$), we use the linear dispersion relation $\omega_p = c_s p$ and get

$$B_{\text{ph}} = \frac{254.9 [1 + n(w_0)] (k_B T)^3}{\hbar^2 c_s^4 \rho_s \lambda_{\min}}. \quad (4.3)$$

On the other hand, for temperatures above 0.6 K, only the portion of the dispersion curve around the roton minimum makes a significant contribution to the integrand in (4.1), then making use of the usual approximations in roton calculations [24], we obtain

$$B_r = \frac{2.079 [1 + n(w_0)] \hbar k_0^3 \sqrt{k_B T}}{c_s^2 \sqrt{\mu} \rho_s \lambda_{\min}}, \quad (4.4)$$

where μ and k_0 are parameters entering the Landau parabolic approximation, $\omega_p = \Delta/\hbar + \hbar(p - k_0)^2/2\mu$.

Experimental determinations of B have been reported only above 1.3 K. However, we may utilize the following expression valid for temperatures below 1 K [1],

$$B = \frac{2D}{\rho_n \kappa} \quad (4.5)$$

and replace D in (4.5) by means of the Iordanskii theoretical estimate for the phonon temperature range [25], yielding

$$B_{\text{ph}} = 8.17 \frac{k_B T}{m_4 c_s^2}. \quad (4.6)$$

On the other hand, for the roton temperature range, we may replace $D = \rho_n v_G \sigma_{\parallel}$ in (4.5) yielding,

$$B_r = \frac{2\sigma_{\parallel}}{\kappa} \sqrt{\frac{2k_B T}{\pi\mu}} \simeq 1.5\sqrt{T} \text{ K}^{-\frac{1}{2}}, \quad (4.7)$$

being v_G the average group velocity of rotons and $\sigma_{\parallel} \simeq 8.38 \text{ \AA}$ the roton scattering length [1, 2]. Notice that we have replaced the friction coefficient D by expressions corresponding to a straight vortex, since corrections due to vortex bending should be negligible, as seen in Sec. I. This of course does not mean that vortices would remain straight against the quasiparticle scattering; on the contrary, thermal excitation of Kelvin waves is undoubtedly expected to occur, although details of this process are not evident from expressions (4.6) and (4.7). However, some features about such a process may be deduced from our results (4.3) and (4.4). First it is convenient to discuss the physical meaning of the frequency w_0 . Since such a frequency is intended for representing the whole spectrum of Kelvin waves (2.3) in the context of an interaction with quasiparticles, it should not be surprising to find that quite distinct values of w_0 could be required in order to better estimate interactions with phonons or rotons. This amounts to assuming a dependence of w_0 on temperature, which may be fully extracted by equating our results with the theoretical expressions (4.6) and (4.7). In fact, from (4.3) and (4.6) we may conclude that phonon scattering at a temperature T should be expected to excite Kelvin waves about a representative frequency given by

$$w_0 \simeq 5.64 \times 10^8 \text{ s}^{-1} \text{ K}^{-3} T^3, \quad (4.8)$$

while (4.4) and (4.7) imply that roton scattering should be likely to excite Kelvin waves about frequency

$$w_0 \simeq 2.08 \times 10^{10} \text{ s}^{-1} \text{ K}^{-1} T. \quad (4.9)$$

Recall that according to the Markov approximation (Sec. III) one should assume $w_0 \gg \Omega_{\text{rot}} \sim 1 \text{ s}^{-1}$, which sets up a lower bound for the validity of the result (4.8) at temperatures above $\sim 0.01 \text{ K}$. In addition, the assumption $\lambda_{\text{min}} \sim 10^3 \text{ \AA}$ implies an upper bound for the Kelvin spectrum, $\max(\omega_k) \sim 10^8 \text{ s}^{-1}$, which turns out to be consistent with a phonon temperature range below 0.4 K in (4.8). However, the values arising from (4.9) seem to be overestimated for roton temperatures, since they would only be consistent with a λ_{min} of order 10^2 \AA . In addition, such values of w_0 could reach the order of cyclotron frequencies arising from some theories of the vortex mass [9], contradicting the assumption $\omega_k \ll \Omega$ of Sec. II A. This suggests that only the qualitative trend $w_0 \sim T$ should be taken into account from the result (4.9).

V. CONCLUDING REMARKS

We have analyzed a microscopic model of mutual friction represented by the dissipative dynamics of a normal fluid flow, which interacts with the helical normal modes of vortices comprising a lattice in thermal equilibrium. Such vortices interact with the quasiparticles forming the normal fluid through a pseudomomentum-conserving scattering Hamiltonian. Assuming a simplified model for the vortex heat bath, we have derived an equation of motion for the quasiparticle pseudomomentum leading to the expected form predicted by the HVBK equations. We have shown that the mutual friction coefficient B turns out to be practically independent of the values of vortex mass arising from diverse theories. Finally, from a comparison of our expression of B with previous theoretical estimates, we have deduced interesting qualitative features about the interaction of quasiparticles with Kelvin modes, namely phonon (roton) scattering at a temperature T should be expected to excite Kelvin waves about representative frequencies proportional to T^3 (T).

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Appendix A: Derivation of the equation of motion for the quasiparticle populations

Our starting point is the Heisenberg equation for the quasiparticle number operator $a_{\mathbf{q}}^\dagger a_{\mathbf{q}}$:

$$\begin{aligned} \frac{d}{dt}(a_{\mathbf{q}}^\dagger a_{\mathbf{q}}) &= \sqrt{\frac{L}{2\hbar\rho_s\kappa}} \sum_{k,\mathbf{p},j} \delta_{p_z q_z} \{ \Lambda_{\mathbf{p}\mathbf{q}} [(p_x - q_x) + i(q_y - p_y)] a_{\mathbf{p}}^\dagger a_{\mathbf{q}} [\alpha_k^{(j)\dagger} + (1 - \delta_{k0})\beta_k^{(j)}] \\ &\quad + \Lambda_{\mathbf{p}\mathbf{q}}^* [(p_x - q_x) - i(q_y - p_y)] [\alpha_k^{(j)} + (1 - \delta_{k0})\beta_k^{(j)\dagger}] a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + \Lambda_{\mathbf{p}\mathbf{q}} [(p_x - q_x) - i(q_y - p_y)] a_{\mathbf{p}}^\dagger a_{\mathbf{q}} [\alpha_k^{(j)} \\ &\quad + (1 - \delta_{k0})\beta_k^{(j)\dagger}] + \Lambda_{\mathbf{p}\mathbf{q}}^* [(p_x - q_x) + i(q_y - p_y)] [\alpha_k^{(j)\dagger} + (1 - \delta_{k0})\beta_k^{(j)}] a_{\mathbf{q}}^\dagger a_{\mathbf{p}} \}, \end{aligned} \quad (\text{A1})$$

where j labels each vortex ($1 \leq j \leq N_v$) and we have taken into account that only the interaction Hamiltonian (3.2) yields a nonvanishing commutator with such an operator. Next we write out the following expressions for the operators appearing on the right-hand side of the above equation, which arise from the formal solutions of the corresponding Heisenberg equations:

$$\begin{aligned} a_{\mathbf{p}}^\dagger(t) a_{\mathbf{q}}(t) &= e^{i(\omega_p - \omega_q)t} a_{\mathbf{p}}^\dagger(0) a_{\mathbf{q}}(0) + i \sqrt{\frac{L}{2\hbar\rho_s\kappa}} \int_0^t d\tau e^{i(\omega_p - \omega_q)\tau} \sum_{k,\mathbf{q}',j} \delta_{p_z q'_z} \Lambda_{\mathbf{q}'\mathbf{p}} \{ [(p_y - q'_y) + i(p_x - q'_x)] \\ &\quad \times [\alpha_k^{(j)\dagger}(t - \tau) + (1 - \delta_{k0})\beta_k^{(j)}(t - \tau)] + [(q'_y - p_y) + i(q'_x - p_x)] [\alpha_k^{(j)}(t - \tau) + (1 - \delta_{k0})\beta_k^{(j)\dagger}(t - \tau)] \} \\ &\quad \times a_{\mathbf{q}'}^\dagger(t - \tau) a_{\mathbf{q}}(t - \tau) - \delta_{q_z q'_z} \Lambda_{\mathbf{q}\mathbf{q}'} \{ [(q'_y - q_y) + i(q'_x - q_x)] [\alpha_k^{(j)\dagger}(t - \tau) + (1 - \delta_{k0})\beta_k^{(j)}(t - \tau)] \\ &\quad + [(q_y - q'_y) + i(q_x - q'_x)] [\alpha_k^{(j)}(t - \tau) + (1 - \delta_{k0})\beta_k^{(j)\dagger}(t - \tau)] \} a_{\mathbf{p}}^\dagger(t - \tau) a_{\mathbf{q}'}(t - \tau), \end{aligned} \quad (\text{A2})$$

$$\alpha_k^{(j)\dagger}(t) = e^{i\Omega t} \alpha_k^{(j)\dagger}(0) + \sqrt{\frac{L}{2\hbar\rho_s\kappa}} \int_0^t d\tau e^{i\Omega\tau} \sum_{\mathbf{p}',\mathbf{q}'} \delta_{p'_z q'_z} \Lambda_{\mathbf{p}'\mathbf{q}'} [(p'_x - q'_x) + i(p'_y - q'_y)] a_{\mathbf{p}'}^\dagger(t - \tau) a_{\mathbf{q}'}(t - \tau) \quad (\text{A3})$$

$$\beta_k^{(j)\dagger}(t) = e^{iw_0 t} \beta_k^{(j)\dagger}(0) + \sqrt{\frac{L}{2\hbar\rho_s\kappa}} \int_0^t d\tau e^{iw_0\tau} \sum_{\mathbf{p}',\mathbf{q}'} \delta_{p'_z q'_z} \Lambda_{\mathbf{p}'\mathbf{q}'} [(p'_x - q'_x) - i(p'_y - q'_y)] a_{\mathbf{p}'}^\dagger(t - \tau) a_{\mathbf{q}'}(t - \tau) \quad (\text{A4})$$

The above expressions and their Hermitian conjugates are then replaced on the right-hand side of (A1), retaining only second-order terms in the scattering amplitudes (weak-coupling approximation). The resulting expression becomes greatly simplified when taking its ensemble average according to the following prescriptions:

(i) All vortex-quasiparticle correlations are neglected, i.e. any average of a product of vortex and quasiparticle operators is approximated by the corresponding product of vortex and quasiparticle separate averages. Such a procedure may be regarded as arising from, (a) an assumption of vanishing initial correlations, i.e. assuming a whole system density operator given by a product of a vortex operator and a quasiparticle operator, and (b) the above weak-coupling approximation by which such vortex-quasiparticle averages should be calculated to zeroth-order in the scattering amplitudes.

(ii) The assumption of a vortex heat bath corresponds to vortex operator averages represented by thermal equilibrium expressions, i.e. $\langle \alpha_k^{(j)\dagger} \alpha_k^{(j)} \rangle = [\exp(\hbar\Omega/k_B T) - 1]^{-1}$, $\langle \beta_k^{(j)\dagger} \beta_k^{(j)} \rangle = [\exp(\hbar w_0/k_B T) - 1]^{-1}$ and $\langle \alpha_k^{(j)\dagger} \rangle = \langle \beta_k^{(j)\dagger} \rangle = 0$.

(iii) We assume that the system is close to equilibrium so that all nondiagonal quasiparticle averages can be neglected, i.e.

$$\langle a_{\mathbf{p}}^\dagger a_{\mathbf{q}} \rangle = \delta_{\mathbf{p}\mathbf{q}} n_{\mathbf{p}}. \quad (\text{A5})$$

Thus, we arrive at the following system of non-Markovian equations for the quasiparticle populations $n_{\mathbf{q}}$:

$$\begin{aligned} \frac{dn_{\mathbf{q}}}{dt} &= \frac{2L^2 N_v}{\hbar \rho_s \kappa \lambda_{\min}} \sum_{\mathbf{p},i} |\Lambda_{\mathbf{p}\mathbf{q}}|^2 \delta_{p_z q_z} (\mathbf{p} - \mathbf{q})^2 \{ \int_0^t d\tau \cos[(\omega_q - \omega_p + w_i)\tau] \{ n_{\mathbf{p}}(t - \tau) [1 + n_{\mathbf{q}}(t - \tau)] \\ &\quad + [n_{\mathbf{p}}(t - \tau) - n_{\mathbf{q}}(t - \tau)] [e^{\hbar w_i/k_B T} - 1]^{-1} \} \\ &\quad - \int_0^t d\tau \cos[(\omega_p - \omega_q + w_i)\tau] \{ n_{\mathbf{q}}(t - \tau) [1 + n_{\mathbf{p}}(t - \tau)] + [n_{\mathbf{q}}(t - \tau) - n_{\mathbf{p}}(t - \tau)] [e^{\hbar w_i/k_B T} - 1]^{-1} \} \}, \end{aligned} \quad (\text{A6})$$

where w_i denotes the frequencies Ω and w_0 . It is easy to verify that (A6) fulfills $\sum_{\mathbf{q}} dn_{\mathbf{q}}/dt = 0$, in agreement with the quasiparticle number-conserving Hamiltonian. It is also instructive to rewrite (A6) in the Markovian limit,

$$\begin{aligned} \frac{dn_{\mathbf{q}}}{dt} = & \frac{2\pi L^2 N_v}{\hbar \rho_s \kappa \lambda_{\min}} \sum_{\mathbf{p}, i} |\Lambda_{\mathbf{p}\mathbf{q}}|^2 \delta_{p_z q_z} (\mathbf{p} - \mathbf{q})^2 \{ \delta(\omega_q - \omega_p + w_i) \{ n_{\mathbf{p}}(t)[1 + n_{\mathbf{q}}(t)] + [n_{\mathbf{p}}(t) - n_{\mathbf{q}}(t)] [e^{\hbar w_i/k_B T} - 1]^{-1} \} \\ & - \delta(\omega_p - \omega_q + w_i) \{ n_{\mathbf{q}}(t)[1 + n_{\mathbf{p}}(t)] + [n_{\mathbf{q}}(t) - n_{\mathbf{p}}(t)] [e^{\hbar w_i/k_B T} - 1]^{-1} \} \}, \end{aligned} \quad (\text{A7})$$

and verify that thermal equilibrium populations, $n_{\mathbf{p}} = [e^{\hbar \omega_p/k_B T} - 1]^{-1}$, lead to a vanishing result on the right-hand side of the above equation.

Appendix B: Study of memory effects

To explore the validity of the Markov approximation, it is convenient to Laplace-transform equation (3.7) according to $\tilde{\mathbf{v}}(z) = \int_0^\infty e^{izt} [\mathbf{v}_n(t) - \mathbf{v}_s] dt$, ($\text{Im } z > 0$). Then one easily finds that

$$\tilde{\mathbf{v}}(z) = [\mathbf{v}_n(0) - \mathbf{v}_s] / [\tilde{\mu}(z) - iz], \quad (\text{B1})$$

where the Laplace transform of the memory kernel can be written as a Cauchy integral:

$$\tilde{\mu}(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\omega \frac{\nu(\omega) + \nu(-\omega)}{\omega - z}, \quad (\text{B2})$$

with,

$$\begin{aligned} \nu(\omega) = & \frac{\hbar L \Omega_{\text{rot}}}{2 \lambda_{\min} k_B T \rho_n \rho_s \kappa^2} \sum_{\mathbf{p}, \mathbf{q}, i} |\Lambda_{\mathbf{p}\mathbf{q}}|^2 \delta_{p_z q_z} \delta(\omega_p - \omega_q + w_i + \omega) \{ |\mathbf{p} - \mathbf{q}|^4 g_+(w_i) + [|\hat{\mathbf{z}} \times (\mathbf{p} \times \hat{\mathbf{z}})|^4 \\ & - |\hat{\mathbf{z}} \times (\mathbf{q} \times \hat{\mathbf{z}})|^4] g_-(w_i) \} n(\omega_q) [1 + n(\omega_p)] [1 + n(w_i)]. \end{aligned} \quad (\text{B3})$$

The poles of $\tilde{\mathbf{v}}(z)$ in (B1) arise from the equation $z = -i\tilde{\mu}(z)$ and the Markov approximation corresponds to the solution $z_M = -i\tilde{\mu}(z \rightarrow i0^+) = -i\nu$ [cf. (3.11)]. A non-Markovian solution can be found iteratively, i.e. we may begin with the Markovian ansatz $z_0 = -i\tilde{\mu}(0)$ and then proceed with $z_1 = -i\tilde{\mu}(z_0)$, $z_2 = -i\tilde{\mu}(z_1)$, and so on. Then the solution z_s is better worked out in terms of the Taylor expansion of $\tilde{\mu}(z)$ around the origin:

$$z_s = -i\nu [1 - i\tilde{\mu}'(0) - \tilde{\mu}''(0)\tilde{\mu}(0)/2 + \dots], \quad (\text{B4})$$

where the second and third terms inside the square brackets represent first- and second-order corrections to the Markov approximation, respectively. The Cauchy integral (B2) and its derivatives in equation (B4) can be written as [20]

$$\tilde{\mu}(0) = \nu(0) \quad (\text{B5a})$$

$$\tilde{\mu}'(0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} [\nu(\omega) + \nu(-\omega) - 2\nu(0)] \quad (\text{B5b})$$

$$\tilde{\mu}''(0) = \nu''(0), \quad (\text{B5c})$$

where the calculation of $\nu(\omega)$ from expression (B3) may be reduced to a single one-dimensional integral (see [24]):

$$\begin{aligned} \nu(\omega) = & \frac{19\hbar^3 \Omega_{\text{rot}}}{140(2\pi)^2 c_s^2 \rho_n \rho_s k_B T \lambda_{\min}} \sum_i [1 + n(w_i)] \int_0^\infty dp |\omega'_p| n(\omega_p + w_i + \omega) [1 + n(\omega_p)] \\ & \times \sum_j \Gamma(p, q_j^{(i)}) g_+(w_i) + \Phi(p, q_j^{(i)}) g_-(w_i), \end{aligned} \quad (\text{B6})$$

where $g_{\pm}(w_i)$ is given by equation (3.9) with

$$\omega_q = \omega_p + w_i + \omega, \quad (\text{B7})$$

$$\Gamma(p, q) = \begin{cases} p^2 q (q^4 + p^4/5 + 2p^2 q^2) & (p \leq q) \\ q^2 p (p^4 + q^4/5 + 2p^2 q^2) & (q \leq p) \end{cases} \quad (\text{B8})$$

$$\Phi(p, q) = \begin{cases} p^2 q(-q^4 + p^4/3 + \frac{2}{3}p^2 q^2) & (p \leq q) \\ q^2 p(p^4 - q^4/3 - \frac{2}{3}p^2 q^2) & (q \leq p) \end{cases} \quad (\text{B9})$$

and $q = q_j^{(i)}$ denote the roots of the equation (B7). The above expressions allow a direct computation of the equations (B5), but it will be more instructive to perform the following dimensional analysis whose results were numerically verified. Firstly, from equations (B5a) and (3.11) we have $\tilde{\mu}(0) = \nu = \Omega_{\text{rot}} B$ which, according to the experimental values of B , must be of order Ω_{rot} at most. Then from equation (B5b) it is not difficult to realize that $\tilde{\mu}'(0)$ should be $\sim \Omega_{\text{rot}}/w_0$ and similarly $\tilde{\mu}''(0) \sim \Omega_{\text{rot}}/w_0^2$. Thus, the first- and second-order corrections in equation (B4) turn out to be, respectively, of order Ω_{rot}/w_0 and $(\Omega_{\text{rot}}/w_0)^2$ which tells us that the Markovian limit corresponds to $\Omega_{\text{rot}} \ll w_0$.

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